

DEFORMATIONS OF REDUCIBLE $\mathrm{SL}(n, \mathbb{C})$ REPRESENTATIONS OF FIBERED 3-MANIFOLD GROUPS

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ABSTRACT. Let M_ϕ be a surface bundle over a circle with monodromy $\phi : S \rightarrow S$. We study deformations of certain reducible representations of $\pi_1(M_\phi)$ into $\mathrm{SL}(n, \mathbb{C})$, obtained by composing a reducible representation into $\mathrm{SL}(2, \mathbb{C})$ with the irreducible representation $\mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{SL}(n, \mathbb{C})$. In particular, we show that under conditions on the eigenvalues of ϕ^* , the reducible representation is contained in a $(n+1+k)(n-1)$ dimensional component of the representation variety, where k is the number of components of ∂M_ϕ . Moreover, the reducible representation is the limit of a path of irreducible representations.

1. INTRODUCTION

Suppose that $S = S_{g,p}$ is a surface of genus g with $p \geq 1$ punctures, where $2g+p > 2$, i.e. S admits a hyperbolic structure. If $\phi : S \rightarrow S$ is a homeomorphism, we can form the mapping torus $M_\phi = S \times [0, 1]/(x, 1) \sim (\phi(x), 0)$. Whenever λ^2 is an eigenvalue of $\phi^* : H^1(S) \rightarrow H^1(S)$ with eigenvector $(a_1, \dots, a_{2g+p-1})^T$ with respect to a generating set $\{[\gamma_1], \dots, [\gamma_{2g+p-1}]\}$ of $H^1(S)$, we obtain a reducible representation $\rho : \pi_1(M_\phi) \rightarrow \mathrm{SL}(2, \mathbb{C})$ by defining,

$$\begin{aligned}\rho_\lambda(\gamma_i) &= \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix} \\ \rho_\lambda(\tau) &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},\end{aligned}$$

where τ is the generator of the fundamental group of the S^1 base of the fiber bundle $S \rightarrow M_\phi \rightarrow S^1$. (Recall that a representation $\rho : G \rightarrow \mathrm{GL}(n, \mathbb{C})$ is *reducible* if the image $\rho(G)$ preserves a proper subspace of \mathbb{C}^n , and otherwise is called *irreducible*.)

When M_ϕ is the complement of a knot K in S^3 , this observation was originally made by Burde [3] and de Rham [4]. Furthermore, the Alexander polynomial is the characteristic polynomial of ϕ^* , so the condition on λ is equivalent to the condition that λ^2 is a simple root of the Alexander polynomial $\Delta_K(t)$. It was shown in [7] that the non-abelian, metabelian, reducible representation ρ_λ is the limit of irreducible representations if λ is a simple root of $\Delta_K(t)$. Recently, Heusener and Medjerrab [6] have shown that the conclusion still holds in $\mathrm{SL}(n, \mathbb{C})$, $n \geq 3$, if ρ_λ is composed with the irreducible representation $r_n : \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{SL}(n, \mathbb{C})$. These results apply

even if the knot complement is not fibered, as long as λ^2 is a simple root of $\Delta_K(t)$.

In this paper, we apply some of the techniques in [6] to show that reducible $\mathrm{SL}(n, \mathbb{C})$ representations of fibered 3-manifolds groups obtained as the composition $\rho_{\lambda, n} = r_n \circ \rho_\lambda$ can be deformed to irreducible representations. If the punctures form a single orbit under ϕ and the complement is the complement of a fibered knot, then the results of [7] and [6] apply. The main result in Theorem 1.1 also covers the cases where M_ϕ is the complement of a fibered link L with $k \geq 2$ components L_1, \dots, L_k , or a k -cusped fibered manifold which is not a link complement. In the statement of Theorem 1.1, $\bar{\phi}$ is the homomorphism on $\bar{S} = S_{g,0}$ obtained from ϕ by filling in the p punctures of $S_{g,p}$. This gives a homeomorphism $\bar{\phi} : \bar{S} \rightarrow \bar{S}$.

Theorem 1.1. *Suppose that λ^2 is a simple eigenvalue of ϕ^* . If $|\lambda| \neq 1$, $\bar{\phi}^* : H^1(\bar{S}) \rightarrow H^1(\bar{S})$ does not have 1 as an eigenvalue, and if for each $2 \leq j \leq n$, we have that λ^{2j} is not an eigenvalue of ϕ^* , then $\rho_{\lambda, n}$ is a limit of irreducible $\mathrm{SL}(n, \mathbb{C})$ representations and is a smooth point of the representation variety $R(\pi_1(M_\phi), \mathrm{SL}(n, \mathbb{C}))$, contained in a unique component of dimension $(n + 1 + k)(n - 1)$.*

When ϕ is a pseudo-Anosov element of the mapping class group, λ is the dilatation factor of ϕ , and the p punctures are exactly the singular points of the invariant foliations of ϕ , ρ_λ is shown to have deformations to irreducible representations under some additional conditions on the eigenvalues of $\bar{\phi}^*$, the map on the closed surface S_g , in [9]. We show that under the same hypotheses, the same holds for $\rho_{\lambda, n}$.

Theorem 1.2. *Suppose that λ^2 is the dilatation of a pseudo-Anosov map ϕ such that the stable and unstable foliations are orientable, and the singular points coincide with the punctures of S . Suppose also that 1 is not an eigenvalue of $\bar{\phi}^*$. Then $\rho_{\lambda, n}$ is a limit of irreducible $\mathrm{SL}(n, \mathbb{C})$ representations and is a smooth point of $R(\pi_1(M_\phi), \mathrm{SL}(n, \mathbb{C}))$, contained in a unique component of dimension $(n + 1 + k)(n - 1)$.*

In Section 2, we give the basic definitions and background about representations of $\mathrm{SL}(2, \mathbb{C})$ into $\mathrm{SL}(n, \mathbb{C})$. Section 3 discusses the general theory of deformations, and Section 4 contains the main results, including relevant cohomological calculations and the irreducibility of nearby representations.

2. REPRESENTATIONS OF $\mathrm{SL}(2, \mathbb{C})$

For notational convenience, we denote $\mathrm{SL}(n) = \mathrm{SL}(n, \mathbb{C})$, $\mathfrak{sl}(n) = \mathfrak{sl}(n, \mathbb{C})$, $\mathrm{GL}(n) = \mathrm{GL}(n, \mathbb{C})$, and $\Gamma_\phi = \pi_1(M_\phi)$. A more general version of the discussion in this section can be found in [6, Section 4].

Let $R = \mathbb{C}[X, Y]$ be the polynomial algebra on two variables. We have an action of $\mathrm{SL}(2)$ on R by,

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X &= dX - bY \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y &= -cX + aY, \end{aligned}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)$. Let $R_{n-1} \subset R$ denote the n -dimensional subspace of homogenous polynomials of degree $n-1$, generated by $X^{l-1}Y^{n-l}$, $1 \leq l \leq n$. The action of $\mathrm{SL}(2)$ leaves R_{n-1} invariant, turning R_{n-1} into a $\mathrm{SL}(2)$ module, and we obtain a representation $r_n : \mathrm{SL}(2) \rightarrow \mathrm{GL}(R_{n-1})$. We can identify R_{n-1} with \mathbb{C}^n by identifying the basis elements $\{X^{l-1}Y^{n-l}\}$ with the standard basis elements $\{e_l\}$ of \mathbb{C}^n . The induced isomorphism turns r_n into a representation $r_n : \mathrm{SL}(2) \rightarrow \mathrm{GL}(n)$, which we will also call r_n . The representation r_n is *rational*, that is the coefficients of the matrix coordinates of $r_n \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are polynomials in a, b, c, d .

We have the following two well-known results about r_n .

Lemma 2.1. [16, Lemma 3.1.3(ii)] *The representation r_n is irreducible.*

Lemma 2.2. [16, Lemma 3.2.1] *Any irreducible rational representation of $\mathrm{SL}(2, \mathbb{C})$ is conjugate to some r_n .*

It is easy to check that r_n maps the unipotent matrices $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ to unipotent elements of $\mathrm{SL}(R_{n-1})$, and the diagonal element $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ is mapped to the diagonal element $\mathrm{diag}(a^{n-1}, a^{n-3}, \dots, a^{-n+1})$. Hence, the image of r_n lies in $\mathrm{SL}(R_{n-1}) \cong \mathrm{SL}(n)$.

We now define $\rho_{\lambda, n} = r_n \circ \rho_\lambda$. As we will only be considering the case when λ is a simple eigenvalue of ϕ^* , and the above lemmas imply the uniqueness of r_n , this gives a well-defined and unique (up to conjugation) representation $\rho_{\lambda, n} : \Gamma_\phi \rightarrow \mathrm{SL}(n)$.

One can also show via explicit calculation that

$$\begin{aligned} r_n \begin{pmatrix} a & a^{-1}b \\ 0 & a^{-1} \end{pmatrix} \cdot X^{l-1}Y^{n-l} &= (a^{-1}X - a^{-1}bY)^{l-1}(aY)^{n-l} \\ &= a^{n-2l+1}(X - bY)^{l-1}Y^{n-l} \\ &= a^{n-2l+1} \sum_{j=0}^{l-1} (-b)^j \binom{l-1}{j} X^{l-j-1}Y^{n-(l-j)}. \end{aligned}$$

In particular, this implies that the space spanned by X^0Y^{n-1} is invariant under the subgroup of upper triangular matrices in $\mathrm{SL}(2)$. Specifically,

$$r_n \begin{pmatrix} a & a^{-1}b \\ 0 & a^{-1} \end{pmatrix} \cdot X^0Y^{n-1} = a^{n-1}X^0Y^{n-1}.$$

As ρ_λ is an upper triangular representation, this action turns R_{n-1} into a Γ_ϕ module, with $\gamma \in \Gamma_\phi$ acting by $r_n \circ \rho_\lambda(\gamma)$. Under this action, $\langle X^0 Y^{n-1} \rangle$ is an invariant submodule.

Definition 2.3. Let $\psi : \Gamma_\phi \rightarrow \mathbb{Z}$ denote the canonical surjection which is dual to the fiber. For a non-zero complex number $\alpha \in \mathbb{C}^*$, we define \mathbb{C}_α to be the Γ_ϕ module \mathbb{C} , where the action of $\gamma \in \Gamma_\phi$ is defined by $x \mapsto \alpha^{\psi(\gamma)}x$.

By the previously defined action of Γ_ϕ , we have that $\langle X^0 Y^{n-1} \rangle$ is isomorphic to $\mathbb{C}_{\lambda^{n-1}}$. Let \bar{R}_{n-1} be the quotient $R_{n-1} / \langle X^0 Y^{n-1} \rangle$. We will need the following facts about the relationship between R_{n-1} , \bar{R}_{n-1} , and $\mathbb{C}_{\lambda^{n-1}}$.

Lemma 2.4. [6, Equations (4.3) and (4.4)] *There are short exact sequences of Γ_ϕ -modules*

$$(2.1) \quad 0 \rightarrow \mathbb{C}_{\lambda^{n-1}} \rightarrow R_{n-1} \rightarrow \bar{R}_{n-1} \rightarrow 1,$$

and,

$$(2.2) \quad 0 \rightarrow R_{n-3} \rightarrow \bar{R}_{n-1} \rightarrow \mathbb{C}_{\lambda^{-n+1}} \rightarrow 0.$$

By composing $\rho_{\lambda,n}$ with the adjoint representation, we also obtain an action of Γ_ϕ on $\mathfrak{sl}(n)$, turning it into a Γ_ϕ module. The following decomposition is a consequence of the Clebsch-Gordan formula (see, for example, [12, Lemma 1.4]).

Lemma 2.5. *With the Γ_ϕ module structure, $\mathfrak{sl}(n) \cong \bigoplus_{j=1}^{n-1} R_{2j}$.*

3. INFINITESIMAL DEFORMATIONS

In this section, let M be a 3-manifold, $\Gamma = \pi_1(M)$, and $\partial\Gamma = \pi_1(\partial M)$. Let $R(\Gamma, \mathrm{SL}(n)) = \mathrm{Hom}(\Gamma, \mathrm{SL}(n))$ be the variety of representations of Γ into $\mathrm{SL}(n)$ and $X(\Gamma, \mathrm{SL}(n)) = R(\Gamma, \mathrm{SL}(n)) / \mathrm{SL}(n)$ be the $\mathrm{SL}(n)$ character variety, where the quotient is the GIT quotient as $\mathrm{SL}(n)$ acts by conjugation.

Suppose $\rho : \Gamma \rightarrow \mathrm{SL}(n)$ is a representation. The group of twisted cocycles $Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$ is defined as the set of maps $z : \Gamma \rightarrow \mathfrak{sl}(n)$ that satisfy the twisted cocycle condition

$$(3.1) \quad z(ab) = z(a) + \mathrm{Ad}_{\rho(a)}z(b),$$

which can be interpreted as the derivative of the homomorphism condition for a smooth family of representation ρ_t at ρ . The derivative of the triviality condition that ρ_t is a smooth family of representations obtained by conjugating ρ gives the coboundary condition,

$$(3.2) \quad z(\gamma) = u - \mathrm{Ad}_{\rho(\gamma)}u,$$

and $B^1(\Gamma; \mathfrak{sl}(n)_\rho)$ is defined as the set of coboundaries, or the cocycles satisfying Equation (3.2). The quotient is defined to be

$$H^1(\Gamma; \mathfrak{sl}(n)_\rho) = Z^1(\Gamma; \mathfrak{sl}(n)_\rho) / B^1(\Gamma; \mathfrak{sl}(n)_\rho).$$

Weil [17, 10] has noted that $Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$ contains the tangent space to $R(\Gamma, \mathrm{SL}(n))$ at ρ as a subspace. The following tools can be used to determine if the representation variety is smooth at ρ , so that we can study the space of cocycles to determine the first order behavior of deformations of a representation ρ . In the following proposition, $C^1(\Gamma; \mathfrak{sl}(n))$ denotes the set of cochains $\{c : \Gamma \rightarrow \mathfrak{sl}(n)\}$.

Proposition 3.1 ([6], Lemma 3.2; [7], Proposition 3.1). *Let $\rho \in R(\Gamma, \mathrm{SL}(n))$ and $u_i \in C^1(\Gamma; \mathfrak{sl}(n)_\rho)$, $1 \leq i \leq j$ be given. If*

$$\rho^j(\gamma) = \exp\left(\sum_{i=1}^j t^i u_i(\gamma)\right) \rho(\gamma)$$

is a homomorphism into $\mathrm{SL}(n, \mathbb{C}[[t]])$ modulo t^{j+1} , then there exists an obstruction class $\zeta_{j+1}^{(u_1, \dots, u_j)} \in H^2(\Gamma; \mathfrak{sl}(n)_\rho)$ such that:

- (1) *There is a cochain $u_{j+1} : \Gamma \rightarrow \mathfrak{sl}(n)$ such that*

$$\rho^{j+1}(\gamma) = \exp\left(\sum_{i=1}^{j+1} t^i u_i(\gamma)\right) \rho(\gamma)$$

is a homomorphism modulo t^{j+2} if and only if $\zeta_{j+1} = 0$.

- (2) *The obstruction ζ_{j+1} is natural, i.e. if f is a homomorphism then $f^* \rho^j := \rho^j \circ f$ is also a homomorphism modulo t^{j+1} and $f^*(\zeta_{j+1}^{(u_1, \dots, u_j)}) = \zeta_{j+1}^{(f^* u_1, \dots, f^* u_j)}$.*

We will apply the previous proposition to the restriction map i^* on cohomology, which is induced by the inclusion map $i : \partial\Gamma \rightarrow \Gamma$. As ∂M_ϕ consists of a disjoint union of tori, we will need to understand $H^1(\pi_1(T^2); \mathfrak{sl}(n)_{r_n \circ \rho})$.

Lemma 3.2. *Suppose $\rho : \pi_1(T^2) \rightarrow \mathrm{SL}(2)$ contains a hyperbolic element in its image. Then $\dim H^1(\pi_1(T^2); \mathfrak{sl}(n)_{r_n \circ \rho}) = 2(n-1)$.*

Proof. Suppose $\gamma \in \pi_1(T^2)$ such that $\rho(\gamma)$ is a hyperbolic element in $\mathrm{SL}(2)$. Then, up to conjugation,

$$\rho(\gamma) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

The image of such an element under the irreducible representation $r_n : \mathrm{SL}(2) \rightarrow \mathrm{SL}(n)$ is conjugate to a diagonal matrix with n distinct eigenvalues. Hence, for any nearby representation $\rho' : \pi_1(T^2) \rightarrow \mathrm{SL}(n)$, $\rho'(\gamma)$ is conjugate to a diagonal matrix with distinct entries. In other words, up to coboundary, we can assume that any class $[z] \in H^1(\pi_1(T^2); \mathfrak{sl}(n)_{r_n \circ \rho})$ has the form of a diagonal matrix $z(\gamma) = \mathrm{diag}(y_1, y_2, \dots, y_n)$ where $\mathrm{tr} z(\gamma) = 0$. Since for any other $\gamma' \in \pi_1(T^2)$, we have that γ' commutes with γ , $z(\gamma')$ must also be diagonal, so the dimension of $H^1(\pi_1(T^2); \mathfrak{sl}(n)_{r_n \circ \rho})$ is $2(n-1)$. \square

Lemma 3.3. *Let M be a 3-manifold with torus boundary components $\partial M = \sqcup_{i=1}^k T_i$. Let $\rho : \pi_1(M) \rightarrow \mathrm{SL}(2)$ be a non-abelian representation such that $\rho(\pi_1(T_i))$ contains a hyperbolic element for each component T_i of ∂M . If $\dim H^1(\Gamma; \mathfrak{sl}(2)_{r_n \circ \rho}) = k(n-1)$ where k is the number of components of ∂M , then $i^* : H^2(M; \mathfrak{sl}(n)_{r_n \circ \rho}) \rightarrow H^2(\partial M; \mathfrak{sl}(n)_{r_n \circ \rho})$ is injective.*

Proof. We have the cohomology exact sequence for the pair $(M, \partial M)$

$$\begin{array}{ccccccc} H^1(M, \partial M) & \longrightarrow & H^1(M) & \xrightarrow{\alpha} & H^1(\partial M) \\ & & \searrow \beta & & \downarrow \\ & & H^2(M, \partial M) & \longrightarrow & H^2(M) \\ & \nearrow i^* & & & \\ & & H^2(\partial M) & \longrightarrow & H^3(M, \partial M) \longrightarrow \end{array}$$

where all cohomology groups are taken to be with the twisted coefficients $\mathfrak{sl}(n)_{r_n \circ \rho}$. A standard Poincaré duality argument [7, 8, 14] gives that α has half-dimensional image. By Lemma 3.2,

$$\dim H^1(\pi_1(T_i); \mathfrak{sl}(n)_{r_n \circ \rho}) = 2(n-1),$$

as long as $\rho(\pi_1(T_i))$ contains a hyperbolic element. Hence, α is injective. Since β is dual to α under Poincaré duality, then β is surjective. This implies that i^* is injective. \square

We now utilize the previous facts to determine sufficient conditions for deforming representations.

Proposition 3.4. *Let M be a 3-manifold with torus boundary components $\partial M = \sqcup_{i=1}^k T_i$. Let $\rho : \Gamma \rightarrow \mathrm{SL}(2)$ be a non-abelian representation such that $\rho(\pi_1(T_i))$ contains a hyperbolic element for each component T_i of ∂M . If $H^1(\Gamma; \mathfrak{sl}(2)_{r_n \circ \rho}) = k(n-1)$ where k is the number of components of ∂M , then $r_n \circ \rho$ is a smooth point of the representation variety $R(\Gamma, \mathrm{SL}(n))$, and it is contained in a unique component of dimension $(n+1+k)(n-1) - \dim H^0(\Gamma; \mathfrak{sl}(n)_{r_n \circ \rho})$.*

Proof. We begin by showing that every cocycle in $Z^1(\Gamma; \mathfrak{sl}(n)_{r_n \circ \rho})$ is integrable.

Suppose we have $u_1, \dots, u_j : \Gamma \rightarrow \mathfrak{sl}(n)$ such that

$$\rho_n^j(\gamma) = \exp\left(\sum_{i=1}^j t^i u_i(\gamma)\right) \rho(\gamma)$$

is a homomorphism modulo t^{j+1} . By Lemma 3.2 and [15], the restriction of ρ_n to $\pi_1(T_i)$ is a smooth point of the representation variety $R(\pi_1(T_i), \mathrm{SL}(n))$. Hence $\rho_n^j|_{\pi_1(T_i)}$ extends to a formal deformation of order $j+1$ by the formal implicit function theorem (see [7], Lemma 3.7). This implies that the restriction of $\zeta_{j+1}^{(u_1, \dots, u_j)}$ to each component $H^2(T_i) < H^2(\partial N_\phi)$ vanishes.

As $H^2(\partial N_\phi) = \oplus_{i=1}^k H^2(T_i)$, hence, $i^* \zeta_{j+1}^{(u_1, \dots, u_j)} = \zeta_{j+1}^{(i^* u_1, \dots, i^* u_j)} = 0$. The injectivity of i^* follows from Lemma 3.3 and implies that $\zeta_{j+1}^{(u_1, \dots, u_j)} = 0$.

Hence, the homomorphism can be extended to a deformation $(r_n \circ \rho)^{j+1}$ of order $j+1$, and inductively to a formal deformation $(r_n \circ \rho)^\infty$.

Applying [7, Proposition 3.6] to the formal deformation $(r_n \circ \rho)^\infty$ results in a convergent deformation. Hence, $r_n \circ \rho$ is a smooth point of the representation variety.

As in [6], we note that the exactness of

$$1 \rightarrow H^0(\Gamma; \mathfrak{sl}(n)_{r_n \circ \rho}) \rightarrow \mathfrak{sl}(n)_{r_n \circ \rho} \rightarrow B^1(\Gamma; \mathfrak{sl}(n)_{r_n \circ \rho})$$

implies that

$$\dim B^1(\Gamma; \mathfrak{sl}(n)_{r_n \circ \rho}) = n^2 - 1 - \dim H^0(\Gamma; \mathfrak{sl}(n)_{r_n \circ \rho}).$$

Thus, we conclude that the local dimension of $R(\Gamma, \mathrm{SL}(n))$ is

$$\dim Z^1(\Gamma; \mathfrak{sl}(n)_{r_n \circ \rho}) = (n+1+k)(n-1) - \dim H^0(\Gamma; \mathfrak{sl}(n)_{r_n \circ \rho}).$$

That it is in a unique component follows from [7, Lemma 2.6]. \square

4. DEFORMING $\rho_{\lambda, n}$

We will now show that $\rho_{\lambda, n}$ satisfies the conditions in Proposition 3.4, so that $\rho_{\lambda, n}$ can be deformed. This will entail a computation of the dimension of the cohomology group $H^1(\Gamma_\phi; \mathfrak{sl}(n)_{\rho_{\lambda, n}})$.

To simplify the computations which follow, we give a presentation of Γ_ϕ with an additional generator γ_{2g+p} . We will choose $\gamma_1, \dots, \gamma_{2g}$ to be standard generators of the fundamental group for the closed surface S_g , and $\gamma_{2g+1}, \dots, \gamma_{2g+p}$ to be curves around the p punctures of S . Then $\pi_1(\Gamma_\phi)$ has a presentation of the form:

$$\langle \gamma_1, \dots, \gamma_{2g+p}, \tau | \tau \gamma_i \tau^{-1} = \phi(\gamma_i), \Pi_{i=1}^g [\gamma_{2i-1}, \gamma_{2i}] = \Pi_{j=1}^p \gamma_{2g+j} \rangle.$$

Up to a choice of generators for $\pi_1(S)$, $\phi^* : H^1(S) \rightarrow H^1(S)$ can be written as a block matrix

$$\begin{pmatrix} [\bar{\phi}^*] & [*] \\ 0 & [P] \end{pmatrix}$$

where $\bar{\phi}^* : H^1(\bar{S}) \rightarrow H^1(\bar{S})$ is the induced map on the first cohomology of the closed surface \bar{S} obtained by filling in the p punctures of S , and $P = (p_{ij})$ is a permutation matrix denoting the permutation of the punctures on S . In particular, $p_{jk_j} = 1$ if and only if $\tau \delta_j \tau^{-1}$ is conjugate to δ_{k_j} , with $p_{jk_j} = 0$ otherwise. We have that $\bar{\phi}^*$ is a symplectic matrix preserving the intersection form ω on \bar{S} . The eigenvalues of P are roots of unity, with 1 occurring as an eigenvalue for each cycle in the permutation.

The following inductive step is based on in [6, Lemma 4.4]. Along with Lemma 2.5, it will allow us to compute the cohomological dimension for arbitrary n from the case when $n = 2$.

Lemma 4.1. *Let $\lambda \in \mathbb{C}^*$ and $n > 3$. Suppose λ^{n-1} is not an eigenvalue of ϕ^* and $\lambda^{n-1} \neq 1$. Then,*

$$H^*(\Gamma_\phi; R_{n-1}) \cong H^*(\Gamma_\phi; R_{n-3}).$$

Proof. The short exact sequence in Equation (2.1) induces a long exact sequence [2, III.6],

$$\begin{aligned} H^k(\pi_1(\Gamma_\phi; \mathbb{C}_{\lambda^{n-1}})) &\rightarrow H^k(\Gamma_\phi; R_{n-1}) \rightarrow H^k(\Gamma_\phi; \bar{R}_{n-1}) \\ &\rightarrow H^{k+1}(\Gamma_\phi; \mathbb{C}_{\lambda^{n-1}}), \end{aligned}$$

which is exact for $k = 0, 1, 2$. Since $\lambda^{n-1} \neq 1$, 0 is the only point of $\mathbb{C}_{\lambda^{n-1}}$ fixed by Γ_ϕ . Consequently, $H^0(\Gamma_\phi; \mathbb{C}_{\lambda^{n-1}}) = 0$. By the universal coefficient theorem,

$$H^1(\Gamma_\phi; \mathbb{C}_{\lambda^{n-1}}) \cong \text{Hom}(\Gamma_\phi; \mathbb{C}; \mathbb{C}_{\lambda^{n-1}}).$$

As λ^{n-1} is not an eigenvalue of ϕ^* , it is also not an eigenvalue of ϕ_* , implying that $H^1(\Gamma_\phi; \mathbb{C}_{\lambda^{n-1}}) = 0$. Since M_ϕ has non-empty boundary and Euler characteristic 0, then it must also follow that $H^2(\Gamma_\phi; \mathbb{C}_{\lambda^{n-1}}) = 0$. Thus, we conclude that

$$H^k(\Gamma_\phi; R_{n-1}) \cong H^k(\Gamma_\phi; \bar{R}_{n-1}),$$

for $k = 0, 1, 2$.

The same argument applied to the short exact sequence in Equation (2.2) yields,

$$H^k(\Gamma_\phi; R_{n-3}) \cong H^k(\Gamma_\phi; \bar{R}_{n-1}),$$

for $k = 0, 1, 2$ since $\lambda^{-(n-1)}$ is an eigenvalue of ϕ^* if and only if λ^{n-1} is an eigenvalue. \square

We now compute the cohomological dimension when $n = 2$. The argument generalizes [9, Theorem 4.1].

Proposition 4.2. *Let $\phi : S \rightarrow S$ be a homeomorphism, with λ^2 a simple eigenvalue of ϕ^* . Suppose also that $|\lambda| \neq 1$ and $\bar{\phi}^* : H^1(\bar{S}) \rightarrow H^1(\bar{S})$ does not have 1 as an eigenvalue. Then $\dim H^1(\Gamma_\phi, \mathfrak{sl}(2)_{\rho_\lambda}) = k$ where k is the number of components of ∂M_ϕ .*

Proof. Let $z \in Z^1(\Gamma_\phi, \mathfrak{sl}(2)_{\rho_\lambda})$. Then z is determined by its values on $\gamma_1, \dots, \gamma_{2g+p}$, and τ , subject to the cocycle condition (3.1) imposed by the relations in Γ_ϕ . These can be computed via the Fox calculus [10, Chapter 3]. Differentiating the relations

$$\tau \gamma_i \tau^{-1} = \phi(\gamma_i),$$

yields

$$\begin{aligned} \frac{\partial[\phi(\gamma_i) \tau \gamma_i^{-1} \tau^{-1}]}{\partial \gamma_i} &= \frac{\partial \phi(\gamma_i)}{\partial \gamma_i} - \phi(\gamma_i) \tau \gamma_i^{-1} = \frac{\partial \phi(\gamma_i)}{\partial \gamma_i} - \tau \\ \frac{\partial[\phi(\gamma_i) \tau \gamma_i^{-1} \tau^{-1}]}{\partial \gamma_j} &= \frac{\partial \phi(\gamma_i)}{\partial \gamma_j}, i \neq j \\ (4.1) \quad \frac{\partial[\phi(\gamma_i) \tau \gamma_i^{-1} \tau^{-1}]}{\partial \tau} &= \phi(\gamma_i) - \phi(\gamma_i) \tau \gamma_i^{-1} \tau^{-1} = \phi(\gamma_i) - 1. \end{aligned}$$

Choosing the basis,

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for $\mathfrak{sl}(2)$, the values $z(\gamma_i)$ can be expressed in coordinates (x_i, y_i, z_i) , where $z(\gamma_i)$ is the matrix

$$z(\gamma_i) = \begin{pmatrix} y_i & x_i \\ z_i & -y_i \end{pmatrix},$$

and we similarly let $z(\tau)$ be given in the coordinates (x_0, y_0, z_0) . The set of coboundaries can be computed from Equation (3.2), as the set of cocycle z' satisfying,

$$\begin{aligned} z'(\gamma_i) &= \begin{pmatrix} -a_i z & 2a_i y + a_i^2 z \\ 0 & a_i z \end{pmatrix} \\ z'(\tau) &= \begin{pmatrix} 0 & x - \lambda x \\ z - \lambda^{-1} z & 0 \end{pmatrix}, \end{aligned}$$

where $x, y, z \in \mathbb{C}$ parametrize $B^1(\Gamma_\phi, \mathfrak{sl}(2)_{\rho_\lambda})$. In particular, adding the appropriate coboundary z' to z , we can assume $x_0 = z_0 = 0$, so that $z(\tau)$ has the form

$$z(\tau) = \begin{pmatrix} y_0 & 0 \\ 0 & -y_0 \end{pmatrix}.$$

We first note that if W is a word in the γ_i , then $\rho(W) = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$ for some real number A . Then, under the chosen basis for $\mathfrak{sl}(2)$, $\rho_\lambda(W)$ acts by

$$\begin{pmatrix} 1 & -2A & -A^2 \\ 0 & 1 & A \\ 0 & 0 & 1 \end{pmatrix}.$$

We obtain one term from $\frac{\partial \phi(\gamma_i)}{\partial \gamma_j}$ for each instance of γ_j in $\phi(\gamma_i)$, and its negation for each instance of γ_j^{-1} in $\phi(\gamma_i)$.

Similarly, we can compute that $\rho_\lambda(\tau)$ acts on $\mathfrak{sl}(2)$ via

$$\begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}.$$

Then z is determined, as in [7], by a vector

$$(x_1, \dots, x_{2g+p}, y_0, y_1, \dots, y_{2g+p}, z_1, \dots, z_{2g+p})^T$$

in the kernel of the matrix

$$S = \begin{pmatrix} \begin{bmatrix} \phi^* - \lambda^2 I \\ 0 \\ 0 \end{bmatrix} & \begin{matrix} -2\lambda a_1 \\ \vdots \\ -2\lambda a_{2g+p} \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{bmatrix} K \\ \phi^* - I \\ 0 \end{bmatrix} & \begin{bmatrix} C \\ D \\ \phi^* - \lambda^{-2} I \end{bmatrix} \end{pmatrix}.$$

As λ^2 is a simple eigenvalue, $\bar{\phi}^*$ is symplectic, and the eigenvalues of P are roots of unity, $\phi^* - \lambda^2 I$ and $\phi^* - \lambda^{-2} I$ have 1 dimensional kernel. Furthermore, since 1 is not an eigenvalue of $\bar{\phi}^*$, $\phi^* - I$ has kernel whose dimension is equal to the number of disjoint cycles of the permutation of the punctures. This is equal to the number of components of ∂M_ϕ . Hence, the kernel of S has dimension at most $2 + k + 1$, where the additional dimension comes from the column vector

$$-2\lambda(a_1, \dots, a_{2g+p}, 0, \dots, 0)^T,$$

in S , and

$$k = \# \text{ of components of } \Sigma = \# \text{ of components of } \partial M_\phi.$$

Consider the upper left portion of the matrix S .

$$U = \begin{pmatrix} \begin{bmatrix} \phi^* - \lambda^2 I \\ 0 \end{bmatrix} & \begin{matrix} -2\lambda a_1 \\ \vdots \\ -2\lambda a_{2g+n} \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{bmatrix} K \\ \phi^* - I \end{bmatrix} \end{pmatrix}.$$

If $\text{null}(S) > 2 + k$, then we must have that $\text{null}(U) > k + 1$.

Since λ^2 is a simple eigenvalue of ϕ^* and $(a_1, \dots, a_{2g})^T$ is an eigenvector of the λ^2 eigenspace, $(a_1, \dots, a_{2g})^T$ is not in the image of $\phi^* - \lambda^2 I$. Hence, for any $y = (y_1, \dots, y_{2g+p})^T$ in the kernel of $\phi^* - I$, there is a unique y_0 such that $Ky - y_0(a_1, \dots, a_{2g})^T$ is in the image of $\phi^* - \lambda I$. Therefore, $\text{null}(U) = k + 1$.

Hence $\text{null}(R) = 2 + k$. However, the solution arising from the kernel of $\phi^* - \lambda^2 I$ is the eigenvector

$$(a_1, \dots, a_{2g+n}, 0, \dots, 0, 0, \dots, 0)^T$$

which is a coboundary. So we have that $\dim H^1(\Gamma_\phi; \mathfrak{sl}(2)_{\rho_\lambda}) \leq k + 1$. Finally, there is one further redundancy since

$$\Pi_{i=1}^g [\gamma_{2i-1}, \gamma_{2i}] = \Pi_{j=1}^p \gamma_{2g+j}.$$

From the $\phi^* - I$ block, we can see that $y_{2g+1}, \dots, y_{2g+p}$ can be freely chosen as long as $y_{2g+j} = y_{2g+k_j}$ whenever γ_{2g+j} and γ_{2g+k_j} are in the same cycle of P . Hence, the upper-left entry of $z(\Pi_{j=1}^n \gamma_{2g+j})$ can be chosen to be any quantity

$$(4.2) \quad y_{2g+1} + y_{2g+2} + \dots y_{2g+p}.$$

The relation $\Pi_{i=1}^g [\gamma_{2i}, \gamma_{2i+1}] = \Pi_{j=1}^p \gamma_{2g+j}$ relates the sum in Equation (4.2) to the upper-left entry of $\Pi_{i=1}^g [\gamma_{2i}, \gamma_{2i+1}]$, which has no dependence on y_{2g+j} , for $1 \leq j \leq p$. This imposes a 1-dimensional relation on the space of cocycles, and we conclude that

$$\dim H^1(\Gamma_\phi, \mathfrak{sl}(2)_{\rho_\lambda}) = k.$$

□

We will need one final technical lemma in order to show that the reducible representation is a limit of irreducible representations. Let ρ_t be a smooth family of representations such that $\rho_0 = \rho_{\lambda, n}$. Since $\rho_{\lambda, n}(\tau)$ is diagonal with distinct eigenvalues, it follows that up to conjugation, ρ_t is diagonal for t sufficiently small. Thus, we can assume that $A(t) = \rho_t(\tau) = \mathrm{diag}(a_{11}(t), a_{22}(t), \dots, a_{nn}(t))$, with $A(0) = \mathrm{diag}(\lambda^{n-1}, \lambda^{n-3}, \dots, \lambda^{-n+1})$. Let $B(t) = \rho_t(\gamma_i)$ for some i such that $a_i \neq 0$. Denote $B(t) = (b_{jl}(t))$. We have that

$$B(0) = \begin{pmatrix} 1 & -a_i \binom{1}{1} & (-a_i)^2 \binom{2}{2} & (-a_i)^3 \binom{3}{3} & \dots & (-a_i)^{n-1} \binom{n-1}{n-1} \\ 0 & 1 & (-a_i) \binom{2}{1} & (-a_i)^2 \binom{3}{2} & \dots & \vdots \\ 0 & 0 & 1 & (-a_i) \binom{3}{1} & \dots & \vdots \\ 0 & 0 & 0 & \ddots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

The following gives a condition for irreducibility of the representation, and is similar to the argument in [1, Proposition 5.4].

Lemma 4.3. *Suppose $A(t)$ and $B(t)$ are matrices as defined above. Suppose also that $b_{n1}^{(n-1)} \neq 0$ and $b_{n1}^{(k)} = 0$ for all $0 < k < n-1$. Then for sufficiently small $t \neq 0$, $A(t)$ and $B(t)$ generate the full matrix algebra $M(n, \mathbb{C})$.*

Proof. Consider the row vectors,

$$(1, 0, \dots, 0)A(0), (1, 0, \dots, 0)B(0), (1, 0, \dots, 0)B^2(0), \dots, (1, 0, \dots, 0)B^{n-1}(0).$$

We have that

$$\begin{aligned} \det \begin{pmatrix} \lambda^{n-1} & 0 & 0 & \dots & 0 \\ 1 & -a_i & (-a_i)^2 & \dots & (-a_i)^{n-1} \\ 1 & -2a_i & (-2a_i)^2 & \dots & (-2a_i)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -(n-1)a_i & (-(n-1)a_i)^2 & \dots & (-(n-1)a_i)^{n-1} \end{pmatrix} \\ = \lambda^{n-1} \det \begin{pmatrix} -a_i & (-a_i)^2 & \dots & (-a_i)^{n-1} \\ -2a_i & (-2a_i)^2 & \dots & (-2a_i)^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -(n-1)a_i & (-(n-1)a_i)^2 & \dots & (-(n-1)a_i)^{n-1} \end{pmatrix}. \end{aligned}$$

Note that the second determinant is 0 if and only if there exist constants c_1, \dots, c_{n-1} , not all equal to 0, such that $f(x) = c_1(-a_i)x + c_2(-a_i)x^2 + \dots + c_{n-1}(-a_i)x^{n-1} = 0$ for $x = 1, \dots, n-1$. But we can also see that $f(0) = 0$, so that $f(x)$ has n roots, so must be identically 0. Hence, it must be that $(1, 0, \dots, 0)A(0)$, $(1, 0, \dots, 0)B(0)$, $(1, 0, \dots, 0)B^2(0)$, \dots , $(1, 0, \dots, 0)B^{n-1}(0)$ are linearly independent, so

$$(1, 0, \dots, 0)A(t), (1, 0, \dots, 0)B(t), (1, 0, \dots, 0)B^2(t), \dots, (1, 0, \dots, 0)B^{n-1}(t)$$

generate \mathbb{C}^n for sufficiently small t .

Now let $g(t)$ be the determinant of the matrix consisting of the column vectors,

$$\begin{aligned} a(t) &= A(t) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11}(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \\ b_j(t) &= B(t)^j \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = B(t)^{j-1} \begin{pmatrix} b_{11}(t) \\ b_{21}(t) \\ \vdots \\ b_{n1}(t) \end{pmatrix}, j = 1, \dots, n-1. \end{aligned}$$

Then,

$$g^{(k)}(t) = \sum_{k_1+k_2+\dots+k_n=k} \frac{k!}{k_1!k_2!\dots k_n!} \det(a^{(k_1)}(t), b_1^{(k_2)}(t), \dots, b_{n-1}^{(k_{n-2})}(t)).$$

Since $b_{n1}^{(k)}(0) = 0$ for $k \leq n-1$, we see that $g^{(k)}(0) = 0$ for $k < (n-1)^2$, and

$$\begin{aligned} \frac{((n-1)!)^{n-1}}{(n^2-2n+1)!} g^{(n^2-2n+1)} \\ = \det(a(0), b^{(n-1)}(0), B(0)b^{(n-1)}(0), \dots, B^{n-2}(0)b^{(n-1)}(0)). \end{aligned}$$

Noting that $B(0) = I + N$ where N is a nilpotent matrix, we can see that if $b_{n1}^{(n-1)} \neq 0$, then

$$\begin{aligned} \frac{((n-1)!)^{n-1}}{(n^2-2n+1)!} g^{(n^2-2n+1)}(0) \\ = \det(a(0), b^{(n-1)}(0), Nb^{(n-1)}(0), \dots, N^{n-2}b^{(n-1)}(0)) \\ \neq 0. \end{aligned}$$

Hence, for sufficiently small $t \neq 0$, we have that $A(t)$ and $B(t)$ generate \mathbb{C}^n .

Let $P_t(x) = (x - a_{22}(t))(x - a_{33}(t)) \cdots (x - a_{nn}(t))$. Then,

$$\frac{P_t(A(t))}{P_t(a_{11}(t))} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes (1, 0, \dots, 0).$$

Since every rank one matrix can be written as $v \otimes w$, and since for any matrix M , we have that $M(v \otimes w) = Mv \otimes w = v \otimes wM$, it follows that $A(t)$ and $B(t)$ generate all rank one matrices for sufficiently small $t \neq 0$. Every matrix is a sum of rank one matrices, then we conclude that $A(t)$ and $B(t)$ generate the full matrix algebra. \square

We now prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.5, $\mathfrak{sl}(n)$ is the direct sum of R_{2j} , $j = 1, \dots, n-1$. The conditions on the eigenvalues of ϕ^* and Lemma 4.1 imply that for each j , $\dim H^1(\Gamma_\phi; R_2) = \dim H^1(\Gamma_\phi; R_{2j})$. By Proposition 4.2, we know that $\dim H^1(\Gamma_\phi; R_2) = k$, hence $H^1(\Gamma_\phi, \mathfrak{sl}(n)_{\rho_{\lambda,n}}) = k(n-1)$. By Proposition 3.4, this implies smoothness of $R(\Gamma_\phi, \mathrm{SL}(n))$ at $\rho_{\lambda,n}$. Since $\rho_{\lambda,n}$ is non-abelian, it has trivial infinitesimal centralizer, so $H^0(\Gamma_\phi; R_2) = 0$, so that the local dimension is $(n+1+k)(n-1)$.

To show that it is the limit of a path of irreducible representations, we note that the $n = 2$ case gives a path of representations ρ_t where $\rho_0 = \rho_\lambda$ and $\rho_t(\gamma_i) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$ satisfies that $c'(0) \neq 0$, since i is chosen so that the i th coordinate of the eigenvector of ϕ^* corresponding to the eigenvalue λ^{-2} is non-zero. A straightforward computation shows that $B(t) = r_n \circ \rho_t(\gamma_i)$ has its $b_{n1}(t)$ coordinate equal to $(-c(t))^{n-1}$, so that the hypotheses of Lemma 4.3 are satisfied. By Burnside's theorem on matrix algebras, it follows that $r_n \circ \rho_t$ is irreducible for sufficiently small $t \neq 0$. \square

We note this result strengthens the conclusions of [6], where it was shown that the image of an irreducible $\mathrm{SL}(2)$ representation under r_n is generically irreducible as an $\mathrm{SL}(n)$ representation. Theorem 1.1 shows that for sufficiently small $t > 0$, a path ρ_t of irreducible representations limits to $\rho_{\lambda,n}$. We obtain the special case in Theorem 1.2 when λ^2 is the dilatation of a pseudo-Anosov map ϕ . When the stable and unstable foliations of ϕ are

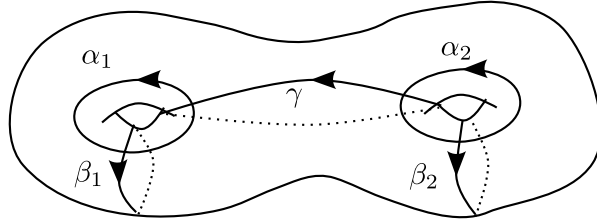


FIGURE 1. The curves $\alpha_1, \alpha_2, \beta_1, \beta_2$ which form the basis for $H_1(S)$, and γ .

orientable, it is a well-known fact that the dilatation is a simple eigenvalue and the largest eigenvalue of ϕ^* (see [5], [11], [13]).

The genus 2 example $\phi : S_{2,2} \rightarrow S_{2,2}$ from [9], obtained from taking the left Dehn twists $T_{\beta_1}, T_{\beta_2}, T_\gamma$, followed by the right Dehn twists $T_{\alpha_1}^{-1}, T_{\alpha_2}^{-1}$, satisfies the hypotheses of Theorem 1.1. Each component of $S_2 \setminus \{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma\}$ contains one of the two punctures. The map on cohomology ϕ^* has two simple eigenvalues $\lambda_1^2 = \frac{5+\sqrt{21}}{2}$ and $\lambda_2^2 = \frac{3+\sqrt{5}}{2}$, along with their reciprocals λ_1^{-2} and λ_2^{-2} . The reducible representations $\rho_{\lambda_i, n}$ are smooth points of $R(\Gamma_\phi, \mathrm{SL}(n))$, each on a component of dimension $(n+3)(n-1)$. There is a two-dimensional family of irreducible representations in $X(\Gamma_\phi, \mathrm{SL}(n))$, which is the image of a two-dimensional family of irreducible representations in $X(\Gamma_\phi, \mathrm{SL}(2))$ under r_n , limiting to $\rho_{\lambda_i, n}$.

The proof of Theorem 1.1 guarantees that these are irreducible, however, it is an interesting question whether there are families of irreducible representations that limit to $\rho_{\lambda, n}$ which are not the image of $\mathrm{SL}(2)$ representations. One can show that when $n > 2$, $b'_{n1}(0) = 0$ for any family of representations ρ_t near $\rho_{\lambda, n}$. An explicit calculation of higher order derivatives of $b_{n1}(t)$ is difficult, but it seems possible that there is a larger family of irreducible representations limiting to $\rho_{\lambda, n}$.

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